

EXISTENCE AND STABILITY OF ALMOST PERIODIC SOLUTIONS OF DIFFERENTIAL EQUATIONS WITH GENERALIZED PIECEWISE CONSTANT ARGUMENT

SAMUEL CASTILLO † AND MANUEL PINTO ‡

ABSTRACT. This work deals with the existence of an almost periodic solution for certain kind of differential equations with generalized piecewise constant argument, almost periodic coefficients which are seen as a perturbation of a linear equation of that kind satisfying an exponential dichotomy on a difference equation. The stability of that solution in a semi-axis studied.

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1. INTRODUCTION

Let \mathbb{N} , \mathbb{Z} , \mathbb{R} , \mathbb{C} be the sets of natural, integer, real and complex numbers, respectively. Denote by $|\cdot|$ the Euclidean norm for every finite dimensional space on \mathbb{R} .

Fix a real valued sequence $(t_n)_{n=-\infty}^{+\infty}$, such that $t_n < t_{n+1}$ and $t_n \rightarrow \pm\infty$ as $n \rightarrow \pm\infty$. For $p \in \mathbb{Z}$, let $\gamma^p : \mathbb{R} \rightarrow \mathbb{R}$ be functions such that $\gamma^p/J_n = t_{n-p}$ for all $n \in \mathbb{Z}$, where $J_n = [t_n, t_{n+1}[$, for all $n \in \mathbb{Z}$.

We are interested in the existence of almost periodic solution of the following linear *Differential Equations with Piecewise Constant Argument Generalized* (DE-PCAG)

$$y'(t) = A(t)y(t) + B(t)y(\gamma^0(t)) + f(t), \quad t \in \mathbb{R} \quad (1)$$

and

$$y'(t) = A(t)y(t) + B(t)y(\gamma^0(t)) + F(t, y_\gamma(t)), \quad t \in \mathbb{R}, \quad (2)$$

where

$$y_\gamma(t) = (y(\gamma^{p_1}(t)), y(\gamma^{p_2}(t)), \dots, y(\gamma^{p_\ell}(t))), \quad (3)$$

where $p_1, p_2, \dots, p_\ell \in \mathbb{N} \cup \{0\}$. DEPCAGs (1) and (2) are seen as perturbation of the linear DEPCAG

$$z'(t) = A(t)z(t) + B(t)z(\gamma^0(t)), \quad (4)$$

where $A, B : \mathbb{R} \rightarrow \mathcal{M}_q(\mathbb{C})$ and $f : \mathbb{R} \rightarrow \mathbb{C}^q$ are locally integrable functions and $F : \mathbb{R} \times W \subseteq \mathbb{R} \times (\mathbb{C}^q)^\ell \rightarrow \mathbb{C}^q$ is a continuous function.

For our study, the following additional assumptions are made.

(H1) A and B are almost periodic functions.

(H2) $(t_n^{(k)})_{n=-\infty}^{+\infty}$, where $t_n^{(k)} = t_{n+k} - t_n$ for all $k \in \mathbb{Z}$, is equipotentially almost periodic for all $k \in \mathbb{Z}$.

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(H3) (H2) holds and for all $\varepsilon > 0$,

$$T(f, \varepsilon) = \left\{ \tau \in \mathbb{R} : |f(t + \tau) - f(t)| \leq \varepsilon, \forall t \in \mathbb{R} - \left(\bigcup_{n \in \mathbb{Z}}]t_n - \varepsilon, t_n + \varepsilon[\right) \right\}$$

is relatively dense and there is $\delta_\varepsilon > 0$ such that $|f(t' + \tau') - f(t')| \leq \varepsilon$ if $\tau' \in \mathbb{R} : |\tau'| \leq \delta_\varepsilon$ and $t', t' + \tau'$ is in some of the intervals $[t_n, t_{n+1}]$.

(H4) F is uniformly almost periodic on W and there is $L > 0$ such that

$$|F(t, x_1, \dots, x_\ell) - F(t, y_1, \dots, y_\ell)| \leq L \sum_{j=1}^{\ell} |x_j - y_j|, \quad (5)$$

for all $t \in \mathbb{R}$ and $(x_1, \dots, x_\ell), (y_1, \dots, y_\ell) \in W$.

A kind of exponential dichotomy is imposed on a part of the linear DEPCAG 4, which will be made explicit in the following section.

This work is motivated by the results in Fink [17, Theorem 7.7, Theorem 8.1 and Theorem 11.31]. Some extensions for piecewise constant argument can be found in [3, 19, 31]. Existence of almost periodic solutions for the impulsive case can be found in [26, 21]. Our focus is to see the almost periodic solutions for DEPCAGs (1) and (2) in terms of the solutions of the difference equation from the Cauchy Operator of the linear part (4), on the points t_n for all $n \in \mathbb{N}$, in the style of [19]. This work is different to Akhmet work [3] where an exponential dichotomy on an ordinary differential system is considered. This work is different to the works on Hong-Yuan [19] and Yuan [31] since a more general y_γ is considered.

Let X be a fundamental matrix of the linear homogeneous system

$$x' = A(t)x \quad (6)$$

and $X(t, s) = X(t)X(s)^{-1}$. Now we follows [4] to say what is the Cauchy matrix for (4).

For $n \in \mathbb{Z}$ and $t \in J_n$ such that $t \geq s$, let $Z_n(t) = X(t, t_n)\mathcal{J}_n(t)$, where $\mathcal{J}_n(t) = I + \int_{t_n}^t X(t_n, u)B(u)du$ and assume that

$$\mathcal{J}_n(t) \text{ is invertible, for all } n \in \mathbb{Z} \text{ and } t \in [t_n, t_{n+1}]. \quad (7)$$

Let

$$H(n) = Z_n(t_{n+1}), \quad (8)$$

for all $n \in \mathbb{Z}$. For $\tau \in \mathbb{R}$, let $k(\tau) \in \mathbb{Z}$ such that $\tau \in J_{k(\tau)}$. Consider $t > s$ such that $k(t) > k(s)$. Then, it is defined

$$Z(t, s) = Z_{k(t)}(t) [H(k(t) - 1)H(k(t) - 2) \cdots H(k(s) + 1)] H(k(s))^{-1} Z_{k(s)}(s)^{-1}. \quad (9)$$

If $t \leq s$, by condition (7), $Z(t, s) = Z(s, t)^{-1}$ is well defined. So, $Z(t, s)$ is the Cauchy matrix for (4). (see [2, 3, 24, 27, 29, 30]).

Consider the difference equation

$$\phi(n + 1) = H(n)\phi(n). \quad (10)$$

Notice that if $z : \mathbb{R} \rightarrow \mathbb{C}$, then $\phi(n) = z(t_n)$ is a solution of (10) if z is a solution of (4).

It will be proved that $H = (H(n))_{n=-\infty}^{+\infty}$ in (8) is almost periodic and that the sequence $h = (h(n))_{n=-\infty}^{+\infty}$, defined by

$$h(n) = \int_{t_n}^{t_{n+1}} X(t_{n+1}, u) f(u) du, \quad (11)$$

for all $n \in \mathbb{Z}$, is almost periodic. Based on the exponential dichotomy of (10) and the almost periodicity of H and h , it will be proved that the bounded solution c of the discrete system

$$c(n+1) = H(n)c(n) + h(n), \quad (12)$$

is almost periodic and the correspondence $h \mapsto c$ is Lipschitz continuous. Then it will be proved that the inhomogeneous linear DEPCAG (1) has an analogous almost periodic solution. The dependence of the almost periodic solution can be seen in terms of the almost periodic solution of the discrete part for (1) and (2), the linear continuous dependence of the almost periodic solution y of (1) in terms of f and the same kind of dependence of c of the almost periodic solution of (12) in terms of h .

By assuming that L in (5) is small enough, an almost periodic solution for the DEPCAG (2) is obtained in terms of the solution of a difference equation. Finally, it will be proved that the almost periodic solution of the DEPCAG (2) is exponentially stable as $t \rightarrow +\infty$ with respect the solutions of (2) for $t \geq 0$. The exponential stability is proved by using a Gronwall inequality on the mentioned difference equation.

The present work is organized as follows: Section 2 provides the main definitions, assumptions and facts that will be used. In the Section 3, the existence of almost periodic solutions for the DEPCAG (1) is studied. In Section 4, that study is extended for the DEPCAG (2) and deals with asymptotic stability for the DEPCAG (2) as $t \rightarrow +\infty$. An example is given in the last section.

2. PRELIMINARIES

(H6) Assume that (10) has an *exponential dichotomy*.

The recent assumption is equivalent to assume that there is a projection $\Pi : \mathbb{C}^q \rightarrow \mathbb{C}^q$ and positive constants ρ, K with $\rho < 1$ such that

$$|\mathcal{G}(n, k)| \leq K \rho^{\pm(n-k)}, \quad (13)$$

for all $n, k \in \mathbb{Z} : \pm(n-k) \leq 0$, where

$$\mathcal{G}(n, k) = \begin{cases} \Phi(n) \Pi \Phi(k+1)^{-1}, & \text{if } n > k \\ -\Phi(n) (I - \Pi) \Phi(k+1)^{-1}, & \text{if } n \leq k \end{cases} \quad (14)$$

and Φ is a fundamental matrix for the system (10). In particular it will be said that system (10) *exponentially stable* as $n \rightarrow +\infty$ if it has an exponential dichotomy with $\Pi = I$.

The recent dichotomy definition has been adapted from that given by Papashinopoulos [20] for (4) when $\gamma = [\cdot]$. It is an exponential dichotomy for (10) which is not obvious to be extended for (4) in terms of $Z(t, s)$ except for cases where the projection for exponential dichotomy commutes with $A(t)$ and $B(t)$. Authors has not found any reference containing a definition of exponential dichotomy for (4).

We start with some classical notions.

It is said that x is a solution of a DEPCAG

$$x'(t) = \tilde{f}(t, x_\gamma(t)), \quad (15)$$

where x_γ is defined in (3), if

- (a) x is continuous on \mathbb{R} ;
- (b) the derivative x' of x exists except possibly at the points $t = t_n$ with $n \in \mathbb{Z}$, where every one-sided derivative exist;
- (c) x is a solution of the DEPCAG (15) except possibly at the points $t = t_n$ with $n \in \mathbb{Z}$.

If \mathbb{E} is a finite dimensional space on \mathbb{R} , $D \subseteq \mathbb{R}$ and $g : D \rightarrow \mathbb{E}$, then $|g|_\infty = \sup_{t \in D} |g(t)|$.

A set $E \subseteq \mathbb{R}$ is called *relatively dense* if there exists a positive real number l such that $E \cap [m, m+l] \neq \emptyset$ for all $m \in \mathbb{R}$. For $\mathbb{A} \subseteq \mathbb{R}$ an additive group and $(\mathbb{E}, |\cdot|)$ a finite dimensional linear space $g : \mathbb{A} \rightarrow \mathbb{E}$ is called *almost periodic* if it is continuous the set of translations $T(g, \varepsilon)$, defined by the set of all $\tau \in \mathbb{A}$ such that $|g(t+\tau) - g(t)| \leq \varepsilon$ for all $t \in \mathbb{A}$, is relatively dense for all $\varepsilon > 0$ (see [17, Definition 1.10]). There will be considered the cases $\mathbb{A} = \mathbb{R}$ (almost periodic functions) and $\mathbb{A} = \mathbb{Z}$ (almost periodic sequences). We can notice by following [21, page 201] that (H3), is a definition of almost periodicity for *piecewise* continuous functions. An alternative definition of almost periodicity for *continuous functions* was given by Salomon Bochner [7] (see Fink [17, page 14] for more detailed reference): A function f is almost periodic if every sequence $(f(t_n + t))_{n=1}^{+\infty}$ of translations of f has a subsequence that converges uniformly for $t \in \mathbb{R}$. A function $F : \mathbb{R} \times W \subseteq \mathbb{R} \times \mathbb{E} \rightarrow \mathbb{E}^q$ is uniformly almost periodic on W , if the set $T(F, \varepsilon, W)$ which denotes the set of all $\tau \in \mathbb{R}$ such that $|F(t+\tau, w) - F(t, w)| \leq \varepsilon$ for all $(t, w) \in \mathbb{R} \times W$, is relatively dense for every $\varepsilon > 0$.

Next, some notations are given. Let $\mathcal{AP}(\mathbb{A}, \mathbb{E})$ be the set of the almost periodic functions from \mathbb{A} into \mathbb{E} . The set $(\mathcal{AP}(\mathbb{A}, \mathbb{C}^q), |\cdot|_\infty)$ is a Banach space.

We say that $\left(t_n^{(k)}\right)_{n=-\infty}^{+\infty}$ is *equipotentially almost periodic*, for all $k \in \mathbb{Z}$ if the set

$$\bigcap_{k \in \mathbb{N}} \left\{ T \in \mathbb{Z} : \left| t_{T+n}^{(k)} - t_n^{(k)} \right| \leq \varepsilon, \text{ for all } n \in \mathbb{Z} \right\}$$

is relatively dense for all $\varepsilon > 0$.

Since A, B are almost periodic, A, B are bounded. Since $\left(t_n^{(k)}\right)_{n=-\infty}^{+\infty}$ is equipotentially almost periodic for all $k \in \mathbb{Z}$, every sequence $\left(t_n^{(k)}\right)_{n=-\infty}^{+\infty}$ is almost periodic for all $k \in \mathbb{Z}$. So, the sequences $\left(t_n^{(k)}\right)_{n=-\infty}^{+\infty}$ are bounded for all $k \in \mathbb{Z}$ (see [21, Theorem 67]) and there exists the positive real number

$$\theta = \sup_{n \in \mathbb{Z}} (t_{n+1} - t_n). \quad (16)$$

Since

$$|Z(t, s)| \leq e^{|A|_\infty(t_{n+1}-t_n)} \left(1 + e^{|A|_\infty(t_{n+1}-t_n)} |B|_\infty(t_{n+1} - t_n) \right),$$

for all $t, s \in J_n$. So, $Z(t, s)$ is bounded. By following [4, 24], we have that $y : \mathbb{R} \rightarrow \mathbb{C}^q$ given by

$$\begin{aligned} y(t) &= Z_{k(t)}(t) \times \left(\sum_{k=-\infty}^{+\infty} \mathcal{G}(k(t), k) \int_{t_k}^{t_{k+1}} X(t_{k+1}, u) f(u) du \right) \\ &+ \int_{\gamma^0(t)}^t X(t, u) f(u) du, \end{aligned} \quad (17)$$

where $t \in \mathbb{R}$, will be the unique bounded solution of (1) which satisfies (H3) (see Theorem 2 below). Moreover, by taking limits $t \rightarrow \gamma^0(t)^+$ and $t \rightarrow \gamma^0(t)^-$, we obtain that y is continuous en every t_n and therefore y is almost periodic.

For $\varepsilon > 0$, let Γ_ε be the set of $r \in \mathbb{R}$ such that there is $k \in \mathbb{Z}$ with

$$\sup_{n \in \mathbb{Z}} \left| t_n^{(k)} - r \right| \leq \varepsilon. \quad (18)$$

Denote by $P_r(\varepsilon)$ the set of all $k \in \mathbb{Z}$ satisfying (18). Let

$$P_\varepsilon = \bigcup_{r \in \Gamma_\varepsilon} P_r(\varepsilon).$$

We will need the following lemmas.

Lemma 1. [21, Lemma 23] *Assume that (H2) holds. Let $\varepsilon > 0$, $\Gamma \subseteq \Gamma_\varepsilon$, $\Gamma \neq \emptyset$ and $P \subseteq \bigcup_{r \in \Gamma} P_r(\varepsilon)$ be such that $P \cap P_r(\varepsilon) \neq \emptyset$ for all $r \in \Gamma$. Then the set Γ is relatively dense if and only if P is relatively dense.*

Lemma 2. [21, Lemma 25] *The following statements are equivalent.*

- (a) (H2) holds;
- (b) The set P_ε is relatively dense for any $\varepsilon > 0$;
- (c) The set Γ_ε is relatively dense for any $\varepsilon > 0$.

Lemma 3. [21, Lemma 29] *Assume that f satisfies (H3). Then $\Gamma_\varepsilon \cap T(f, \varepsilon)$ is relatively dense.*

By mean standard arguments, it can be proved the following result.

Lemma 4. (a) *If f_1, f_2 are functions satisfying (H3), then given $\varepsilon > 0$,*

$\Gamma_\varepsilon \cap T(f_1, \varepsilon) \cap T(f_2, \varepsilon)$ is relatively dense.

- (b) *If $(g_1(n))_{n=-\infty}^{+\infty}$ and $(g_2(n))_{n=-\infty}^{+\infty}$ are almost periodic solutions, then given $\varepsilon > 0$, $P_\varepsilon \cap T(g_1, \varepsilon) \cap T(g_2, \varepsilon)$ is relatively dense.*

For the following results, remind that q is the dimension of DEPCAG (4). Notice that they depends only on the assumptions (H1) and (H3).

Lemma 5. *Consider θ defined in (16). Let $K_0 = \exp(|A|_\infty \theta)$, $K_1 = \sup_{n \in \mathbb{Z}} e^{\left(|A|_\infty \left| t_{n+1}^{(p)} - \tau \right| \right)}$*

and $K_2 = K_0 K_1$. Then

- (a) *$|X(t, s)| \leq \sqrt{q} K_0$, for all $t, s \in \mathbb{R}$ such that $|s - t| \leq \theta$;*

- (b) *If $\tau > 0$, $p \in \mathbb{N}$ and $u \in [t_n, t_{n+1}]$ then*

$$\begin{aligned} |X(t_{n+p+1}, u + \tau) - X(t_{n+1}, u)| &\leq \sqrt{q} \cdot \left[K_1 |A|_\infty \left| t_n^{(p)} - \tau \right| \right. \\ &+ K_2 |A(\cdot + \tau) - A(\cdot)|_\infty |t_{n+1} - t_n| \\ &\times \exp(|A|_\infty (t_{n+1} - t_n)) \left. \right]; \end{aligned}$$

(c) If $\tau > 0$, $p \in \mathbb{N}$ and $t \in [t_n, t_{n+1}]$ then

$$\begin{aligned} |X(t + \tau, t_{n+p}) - X(t, t_n)| &\leq \sqrt{q} \cdot \left[K_1 \left| t_n^{(p)} - \tau \right| \right. \\ &\quad + K_2 |A(\cdot + \tau) - A(\cdot)|_\infty \left| t_{n+1}^{(p)} - \tau - (t_{n+1} - t_n) \right| \\ &\quad \times \exp \left(|A|_\infty \left(\left| t_n^{(p)} - \tau \right| + \theta \right) \right); \end{aligned}$$

(d) If $\tau > 0$ and $t, s \in \mathbb{R} : |t - s| \leq \theta$ then

$$|X(t + \tau, s + \tau) - X(t, s)| \leq \sqrt{q} K_0 |A(\cdot + \tau) - A(\cdot)|_\infty;$$

(e) If $\tau > 0$, $p \in \mathbb{N}$ and $u \in [t_n, t_{n+1}]$ then

$$\begin{aligned} |X(t_{n+p+1}, t_{n+p}) - X(t_{n+1}, t_n)| &\leq K_2 |X(u + \tau, t_{n+p}) - X(u, t_n)| \\ &\quad + \sqrt{q} K_0 |X(t_{n+p+1}, u + \tau) - X(t_{n+p}, u)|. \end{aligned}$$

Proof: Part (a) follows immediately. To prove (b), assume without loss of generality that $t_{n+p+1} - \tau \geq t_{n+1}$. Notice that for $u \in [t_n, t_{n+1}]$,

$$\begin{aligned} \Delta_n(u) &= \int_{t_{n+1}}^{t_{n+p+1}-\tau} X(t_{n+p+1}, \xi + \tau) A(\xi + \tau) d\xi \\ &\quad + \int_u^{t_{n+1}} X(t_{n+p+1}, \xi + \tau) [A(\xi + \tau) - A(\xi)] d\xi \\ &\quad + \int_u^{t_{n+1}} \Delta_n(\xi) A(\xi) d\xi, \end{aligned}$$

where $\Delta_n(u) = X(t_{n+p+1}, u + \tau) - X(t_{n+1}, u)$. Then

$$\begin{aligned} |\Delta_n(u)| &\leq \int_{t_{n+1}}^{t_{n+p+1}-\tau} |X(t_{n+p+1}, \xi + \tau)| |A(\xi + \tau)| d\xi \\ &\quad + \int_{t_n}^{t_{n+1}} |X(t_{n+p+1}, \xi + \tau)| |A(\xi + \tau) - A(\xi)| d\xi \\ &\quad + \int_u^{t_{n+1}} |\Delta_n(\xi)| |A(\xi)| d\xi. \end{aligned}$$

So, by Gronwall's inequality the result is obtained.

Similarly, assume without loss of generality that $t_{n+1} \geq t_{n+p} - \tau$. If $\Delta_n^*(t) = X(t + \tau, t_{n+p}) - X(t, t_n)$, then

$$\begin{aligned} |\Delta_n^*(t)| &\leq \left| \int_{t_{n+p}-\tau}^{t_n} |A(\xi)| |X(\xi + \tau, t_{n+p})| d\xi \right| \\ &\quad + \int_{t_{n+p}-\tau}^{t_{n+1}} |A(\xi + \tau) - A(\xi)| |X(\xi + \tau, t_{n+p})| d\xi \\ &\quad + \int_{t_n}^t |A(\xi)| |\Delta_n^*(\xi)| d\xi, \end{aligned}$$

for $t \in [t_n, t_{n+1}]$. So, by Gronwall's inequality, (c) is obtained. To prove part (d), proceed as in the proof of [19, Proposition 8]. To prove (e), notice that

$$\begin{aligned} X(t_{n+p+1}, t_{n+p}) - X(t_{n+1}, t_n) &= X(t_{n+p+1}, u + \tau)[X(u + \tau, t_{n+p}) - X(u, t_n)] \\ &\quad + [X(t_{n+p+1}, u + \tau) - X(t_{n+p}, u)]X(u, t_n) \end{aligned}$$

and apply the previous results. \square

By Lemma 5, the following result is obtained.

Lemma 6. *Consider θ defined in (16). Let $\varepsilon > 0$, $\tau \in \Gamma_\varepsilon \cap T(A, \varepsilon)$ and $p \in P_\tau(\varepsilon)$. Then there is $K' > 0$ such that for all $n \in \mathbb{Z}$*

- (a) $|X(t_{n+p+1}, u + \tau) - X(t_{n+1}, u)| \leq K'\varepsilon$, for all $u \in [t_n, t_{n+1}]$;
- (b) $|X(t + \tau, t_{n+p}) - X(t, t_n)| \leq K'\varepsilon$, for all $t \in [t_n, t_{n+1}]$;
- (c) $|X(t + \tau, s + \tau) - X(t, s)| \leq K'\varepsilon$, for all $s, t \in \mathbb{R} : |t - s| \leq \theta$;
- (d) $|X(t_{n+p+1}, t_{n+p}) - X(t_{n+1}, t_n)| \leq K'\varepsilon$.

Notice that the existence of $p \in P_\tau(\varepsilon)$ is given by Lemma 2 and the existence of $\tau \in \Gamma_\varepsilon \cap T(A, \varepsilon)$ is given by Lemma 3.

Lemma 7. *The sequence $H = (H(n))_{n=-\infty}^{+\infty}$ given by (8) and the sequence $h = (h(n))_{n=-\infty}^{+\infty}$ given by (11) are almost periodic.*

Proof: Firstly, notice that $H(n) = X(t_{n+1}, t_n) + \psi(n)$, for all $n \in \mathbb{Z}$, where

$$\psi(n) = \int_{t_n}^{t_{n+1}} X(t_{n+1}, u)B(u)du.$$

From Lemma 6 (d), it is not hard to see that $(X(t_{n+1}, t_n))_{n=-\infty}^{+\infty}$ is almost periodic. ψ is also almost periodic. In fact, let $\varepsilon > 0$. From Lemma 4, $\Gamma = T(A, \varepsilon) \cap T(B, \varepsilon) \cap \Gamma_\varepsilon$ is relatively dense. Let $p \in P = \bigcup_{\tau \in \Gamma} P_\tau(\varepsilon)$, so there is $\tau \in \Gamma$

such that $p \in P_\tau(\varepsilon)$. Then, for all $n \in \mathbb{Z}$ it is obtained

$$\begin{aligned} \psi(n+p) - \psi(n) &= \int_{t_{n+p}}^{t_{n+p+1}} X(t_{n+p+1}, u)B(u)du - \int_{t_n}^{t_{n+1}} X(t_{n+1}, u)B(u)du \\ &= \int_{t_{n+p}}^{t_{n+p+1}} X(t_{n+p+1}, u)B(u)du - \int_{t_n+\tau}^{t_{n+p+1}} X(t_{n+p+1}, u)B(u)du \\ &\quad + \int_{t_n+\tau}^{t_{n+p+1}} X(t_{n+p+1}, u)B(u)du - \int_{t_n}^{t_{n+1}} X(t_{n+p+1}, u + \tau)B(u + \tau)du \\ &\quad + \int_{t_n}^{t_{n+1}} X(t_{n+p+1}, u + \tau)B(u + \tau)du - \int_{t_n}^{t_{n+1}} X(t_{n+1}, u)B(u)du, \\ &= \int_{t_{n+p}}^{t_{n+p+1}} X(t_{n+p+1}, u)B(u)du + \int_{t_{n+1}+\tau}^{t_{n+p+1}} X(t_{n+p+1}, u)B(u)du \\ &\quad + \int_{t_n}^{t_{n+1}} [X(t_{n+p+1}, u + \tau)B(u + \tau) - X(t_{n+1}, u)B(u)] du. \end{aligned}$$

By Lemmas 5 and 6, there are positive constants M and K' such that

$$\begin{aligned} |\psi(n+p) - \psi(n)| &\leq \left| t_n^{(p)} - \tau \right| M + \left| t_{n+1}^{(p)} - \tau \right| M + K'\varepsilon, \\ &\leq [2M + K']\varepsilon \end{aligned}$$

for all $n \in \mathbb{Z}$. So, $p \in T(\psi, [2M + K']\varepsilon)$. Since p was taken arbitrarily in P , $P \subseteq T(\psi, [2M + K']\varepsilon)$. By Lemma 1, P is relatively dense. So, $T(\psi, [2M + K']\varepsilon)$ is relatively dense. Since $\varepsilon > 0$ is arbitrary, ψ is almost periodic. Therefore, $H = (H(n))_{n=-\infty}^{+\infty}$ is almost periodic.

In the similar way, h is almost periodic. \square

3. INHOMOGENEOUS LINEAR DEPCAG

To study the existence of an almost periodic solution of the DEPCAG (1), it is reminded that $f \in \mathcal{AP}(\mathbb{R}, \mathbb{C}^q)$.

By mean of the constant variation formula [4, 24],

$$y(t) = Z(t, k(t))c(k(t)) + \int_{\gamma^0(t)}^t X(t, u)f(u)du, \quad (19)$$

is obtained, for all $t \in \mathbb{R}$, where c is solution of the discrete system (12). By taking $t \rightarrow t_{n+1}^-$, it is obtained a solution y for (1) such that $y(t_n) = c(n)$ for all $n \in \mathbb{Z}$. It will be proved that y is almost periodic.

If c is the bounded solution of equation (12) then

$$c(n) = \sum_{k=-\infty}^{+\infty} \mathcal{G}(n, k)h(k), \quad (20)$$

where the Green matrix $\mathcal{G}(n, k)$ is given by (14) and h is given by (11).

From (19) and (20), y is the bounded solution of (1) and satisfies (17). This relation shows y as a bounded linear function of f .

By using the equivalent definition of almost periodicity due to S. Bochner, two important facts are obtained.

Lemma 8. ([19, Proposition 7] and [32]) *A sequence $x = (x(n))_{n=-\infty}^{+\infty}$ is almost periodic if and only if for any integer sequences $(k'_j)_{j=1}^{+\infty}$ and $(\ell'_j)_{j=1}^{+\infty}$ there are subsequences $k = (k_j)_{j=1}^{+\infty}$ and $\ell = (\ell_j)_{j=1}^{+\infty}$ of $(k'_j)_{j=1}^{+\infty}$ and $(\ell'_j)_{j=1}^{+\infty}$ respectively, such that*

$$T_k T_\ell x = T_{k+\ell} x,$$

uniformly on \mathbb{Z} , where $k + \ell = (k_j + \ell_j)_{j=1}^{+\infty}$, $T_m x(n) = \lim_{j \rightarrow +\infty} x(n + m_j)$ and $m = (m_j)_{j=1}^{+\infty} \in \{k, \ell, k + \ell\}$, for all $n \in \mathbb{Z}$.

Theorem 1. *Assume that hypotheses (H1), (H3) and (H6) are satisfied. If c is given by (20), then c is the unique almost periodic solution of the linear inhomogeneous difference system (12). Moreover,*

$$|c|_\infty \leq \frac{2K}{1-\rho} |h|_\infty. \quad (21)$$

Proof: By Lemmas 7, ?? and 8, for any integer sequences $(k'_j)_{j=1}^{+\infty}$ and $(\ell'_j)_{j=1}^{+\infty}$ there are subsequences $k = (k_j)_{j=1}^{+\infty}$ and $\ell = (\ell_j)_{j=1}^{+\infty}$ of $(k'_j)_{j=1}^{+\infty}$ and $(\ell'_j)_{j=1}^{+\infty}$ respectively, such that $T_{k+\ell} H = T_k T_\ell H$ and $T_{k+\ell} h = T_k T_\ell h$, uniformly on \mathbb{Z} .

Now, notice that c given by (20) is the only solution of (12) which is bounded. Moreover, $z = T_{k+\ell}c$ and $z = T_k T_\ell c$ are bounded solutions of

$$z(n+1) = T_{k+\ell}H(n)z(n) + T_{k+\ell}h(n),$$

$$z(n+1) = T_k T_\ell H(n)z(n) + T_k T_\ell h(n),$$

respectively. By uniqueness $T_{k+\ell}c = T_k T_\ell c$. So, $c = (c(n))_{n=-\infty}^{+\infty}$ is an almost periodic sequence. Since c is given by (20), it is the only bounded solution of (12) and satisfies (21). \square

Then the following result is obtained.

Theorem 2. *Consider θ defined in (16). Assume that hypotheses (H1), (H3) and (H6) are satisfied. Then, DEPCAG (1) has a unique almost periodic solution. Moreover,*

$$|y|_\infty \leq K_3 |f|_\infty, \quad (22)$$

$$\text{where } K_3 = \left[\sqrt{q}K_0(1 + |B|_\infty\theta) \frac{2K}{1-\rho} + 1 \right] \sqrt{q}K_0\theta.$$

Proof: Let $\varepsilon > 0$. By Lemma 4, there is $\tau \in T(A, \varepsilon) \cap T(B, \varepsilon) \cap T(f, \varepsilon)$ and $p \in P_\varepsilon \cap T(c, \varepsilon)$. Let y be the solution of (1). Fix $t \in \mathbb{R}$ and let $n \in \mathbb{Z}$ such that $t \in J_n$. Then,

$$\begin{aligned} y(t+\tau) - y(t) &= [X(t+\tau, t_{n+p}) - X(t, t_n)]c(n+p) \\ &+ X(t, t_n)[c(n+p) - c(n)] \\ &+ \int_{t_{n+p}-\tau}^t [X(t+\tau, u+\tau) - X(t, u)]B(u+\tau)du \cdot c(n+p) \\ &+ \int_{t_{n+p}-\tau}^t X(t, u)B(u+\tau)du \cdot [c(n+p) - c(n)] \\ &+ \int_{t_{n+p}-\tau}^t X(t, u)[B(u+\tau) - B(u)]du \cdot c(n) \\ &+ \int_{t_{n+p}-\tau}^{t_n} X(t, u)B(u)du \cdot c(n) \\ &+ \int_{t_{n+p}-\tau}^t [X(t+\tau, u+\tau) - X(t, u)]f(u+\tau)du \\ &+ \int_{t_{n+p}-\tau}^t X(t, u)[f(u+\tau) - f(u)]du \\ &+ \int_{t_{n+p}-\tau}^{t_n} X(t, u)f(u)du \end{aligned}$$

So, by Lemmas 5 and 6, there is $K' > 0$ large enough such that $|y(t+\tau) - y(t)| \leq \varepsilon K'$ for all $t \in \mathbb{R}$. Since $\tau > 0$ was taken arbitrarily in $T(A, \varepsilon) \cap T(B, \varepsilon) \cap T(f, \varepsilon)$, this set is contained in $T(x, \varepsilon K')$. By Lemma 4, $T(x, \varepsilon K')$ is relatively dense. Since $\varepsilon > 0$ was taken arbitrarily, y is an almost periodic solution of (1). From (17), it can be noticed that y is the unique bounded solution of DEPCAG (1). So, y is the unique almost periodic solution of DEPCAG (1).

Since $Z(t, s)$ is bounded and the relations (16), (17) and (21) are satisfied, we have inequality (22). \square

4. THE NONLINEAR DEPCAG (2)

To study the existence of an almost periodic solution of the DEPCAG (2), it is reminded that $W \subseteq (\mathbb{C}^q)^\ell$ is not empty and the set

$$T(F, \varepsilon, W) = \{\tau \in \mathbb{R} : |F(t + \tau, w) - F(t, w)| \leq \varepsilon, \text{ for all } (t, w) \in \mathbb{R} \times W\}$$

is relatively dense for all $\varepsilon > 0$.

Lemma 9. *Let $y : \mathbb{R} \rightarrow \mathbb{C}^q$ an almost periodic function. Assume that (H2) is satisfied and F satisfies (H4). Then $F(t, y_\gamma(t))$ satisfies (H3).*

Proof: Let $\varepsilon > 0$ and $\tau \in T(y, \varepsilon) \cap T(F, \varepsilon, W)$. Since y is almost periodic, it is uniformly continuous. So, there is $\delta > 0$ such that $s, t \in \mathbb{R} : |s - t| \leq \delta$ implies that $|y(t) - y(s)| \leq \varepsilon$. Since $P_\tau(\delta) \neq \emptyset$, $|\gamma^{p_j}(t + \tau) - (\gamma^{p_j}(t) + \tau)| \leq \delta$, for $j = 1, \dots, \ell$. Moreover,

$$\begin{aligned} |F(t + \tau, y_\gamma(t + \tau)) - F(t, y_\gamma(t))| &\leq |F(t + \tau, y_\gamma(t + \tau)) - F(t, y_\gamma(t + \tau))| \\ &\quad + |F(t, y_\gamma(t + \tau)) - F(t, y_\gamma(t))| \\ &\leq \varepsilon + L\ell\varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ was taken arbitrarily, $F(t, y_\gamma(t))$ satisfies (H3). \square

Then, the following result is obtained.

Theorem 3. *Assume that (H1), (H2) and (H6) hold. Assume that F satisfies (H4). If*

$$2 \frac{KL\ell}{1 - \rho} < 1, \quad (23)$$

then equation (2) has an almost periodic solution.

Proof: Let

$$(\mathcal{T}c)(n) = \sum_{k=-\infty}^{+\infty} \mathcal{G}(n, k)h(k, \hat{c}(k)), \quad (24)$$

where $h(n, \hat{c}(n)) = \int_{t_n}^{t_{n+1}} X(t_{n+1}, s)F(s, \hat{c}(n))ds$ and $\mathcal{G}(n, k)$ is given in (14) and $\hat{c}(n) = (c(n - p_1), \dots, c(n - p_\ell))$.

If c is a fixed point of the operator defined by (24) then c is solution of the difference equation

$$c(n + 1) = H(n)c(n) + h(n, \hat{c}(n)). \quad (25)$$

If c is almost periodic then $h(n, \hat{c}(n))$ is almost periodic. In that case, $\mathcal{T}c$ is almost periodic. So, $\mathcal{T}(\mathcal{AP}(\mathbb{Z}, \mathbb{C}^q)) \subseteq \mathcal{AP}(\mathbb{Z}, \mathbb{C}^q)$. Moreover,

$$|(\mathcal{T}c_1)(n) - (\mathcal{T}c_2)(n)| \leq 2 \frac{KL\ell}{1 - \rho} |c_1 - c_2|_\infty.$$

If (23) holds, $\mathcal{T} : \mathcal{AP}(\mathbb{Z}, \mathbb{C}^q) \rightarrow \mathcal{AP}(\mathbb{Z}, \mathbb{C}^q)$ is a contracting mapping. By the Banach fixed point theorem, there is $c \in \mathcal{AP}(\mathbb{Z}, \mathbb{C}^q)$ a unique fixed point for \mathcal{T} .

Therefore, equation (25) has an almost periodic solution c . By Theorem 2, it can be constructed a solution y of (2) which is almost periodic. \square

From now on we will be devoted, to the exponential stability of the almost periodic solution of (2) whose existence was proved in the previous section. So, we say what we will understand by exponential stability.

Assume that $p_j > 0$ for $j = 1, \dots, \ell$. Let $p = \max_{j=1, \dots, \ell} p_j$.

A solution y of the DEPCAG (2), is *exponentially stable* as $t \rightarrow +\infty$ if there is $\alpha \in]0, 1[$ such that given $\varepsilon > 0$, there exists $\delta > 0$ such that $\tilde{y} = \tilde{y}(t)$ is a solution of (2) defined for $t \geq t_0$ then

$$\max_{j=0,1,\dots,p} |y(t-j) - \tilde{y}(t-j)| \leq \delta$$

implies

$$|\tilde{y}(t) - y(t)| \leq \varepsilon \alpha^t, \text{ for all } t \geq t_0. \quad (26)$$

This kind of stability is in the half axis although the solution being exponentially stable is defined on the whole axis.

The recent definition is independent on the choice of t_0 . Any other value could be chosen.

Let $\Phi(n, k) = \Phi(n)\Phi(k)^{-1}$, for all $(n, k) \in \mathbb{Z}^2$. Assume that the difference system (10) is *exponentially stable* as $n \rightarrow +\infty$, i.e., assume that there are positive constants ρ, K with $\rho < 1$ and $K \geq 1$ such that

$$|\Phi(n, k+1)| \leq K \rho^{n-k}, \quad (27)$$

for all $n, k \in \mathbb{Z} : n \geq k$.

By Theorem 3 and the exponential stability, the condition

$$\frac{KL\ell}{1-\rho} < 1, \quad (28)$$

insures the existence of a unique almost periodic solution $y = y(t)$ of DEPCAG (2) defined for all $t \in \mathbb{R}$.

For DEPCAG (4), notice that an exponential stability for (10) implies a direct notion on exponential stability on $Z(t, s)$. In fact, from (9) and (27), it is obtained, for $n > k$, $t \in J_n$ and $s \in]t_k, t_{k+1}]$, that

$$|Z(t, s)| \leq K_4 \rho^{n-k},$$

where $K_4 = K\sqrt{q}K_0^2[1 + \sqrt{q}K_0|B|_\infty\theta]^2$ and θ is given in (16). Since $t - s \leq t_{n+1} - t_k \leq \theta(n - k + 2)$,

$$|Z(t, s)| \leq K_4 \rho^{-2} \rho^{\frac{t-s}{\theta}}.$$

If $\eta_0, \eta_1, \dots, \eta_p \in \mathbb{C}^q$, it is not hard to see that the difference system (25) has a solution $\tilde{c} = \tilde{c}(n)$ defined for $n \geq 0$ with the initial conditions $\tilde{c}(-j) = \eta_j \in \mathbb{C}^q$ for $j = 0, 1, \dots, p$.

Let

$$\begin{aligned} \tilde{y}(t) &= Z_{k(t)}(t) \cdot (\Phi(n, 0)\tilde{c}(0) \\ &+ \sum_{k=0}^{n-1} \Phi(n, k+1) \int_{t_k}^{t_{k+1}} X(t_{k+1}, u) F(u, \tilde{c}(k-p_1), \dots, \tilde{c}(k-p_\ell)) du \\ &+ \int_{\gamma_0(t)}^t X(t, u) F(u, \tilde{c}(n-p_1), \dots, \tilde{c}(n-p_\ell)) du, \end{aligned} \quad (29)$$

where $t \geq t_0$. Then, $\tilde{y} = \tilde{y}(t)$ is the unique solution of (2) with $t \geq t_0$ and fixed initial conditions $\tilde{y}(t-j) = \eta_j$ for $j = 0, 1, \dots, p$.

Then, the following result is given.

Theorem 4. *Assume that (H1), (H2), (H4) hold and that the difference system (10) has an exponential stability as $n \rightarrow +\infty$. Assume that (27) and (28) hold. If y is the almost periodic solution of (2) and \tilde{y} is solution of (2) for $t \geq t_0$ with initial conditions $\tilde{y}(t_{-j}) = \eta_j$ for $j = 0, 1, \dots, p$, then there is $\tilde{K} > 0$ such that*

$$|y(t) - \tilde{y}(t)| \leq \tilde{K} (\rho(1 + KL\ell\rho^{-p}))^n \max_{j=0,1,\dots,p} |c(-j) - \eta_j|, \quad (30)$$

where $t \geq t_0$. Hence, if

$$\frac{KL\ell}{1 - \rho} < \rho^{p-1} \quad (31)$$

then y is exponentially stable.

Proof: Consider that $c(n) = y(t_n)$ and $\tilde{c}(n) = \tilde{y}(t_n)$ for all integer $n \geq n_0$. Let $u(n) = c(n) - \tilde{c}(n)$ for all $n \in \mathbb{Z}$. Then, for $n_0 \in \mathbb{Z}$,

$$\begin{aligned} |u(n)| &\leq |\Phi(n, 0)| |u(0)| + \sum_{k=0}^{n-1} |\Phi(n, k+1)| |F(k, \hat{c})(n) - F(k, \tilde{c}(n))| \\ &\leq K\rho^n |u(0)| + KL \sum_{k=0}^{n-1} \rho^{n-k} \sum_{j=1}^{\ell} |u(k - p_j)|. \end{aligned}$$

Let $\omega(n) = \rho^{-n} \sum_{j=1}^{\ell} |u(n - p_j)|$ and $v(n) = \rho^{-n} |u(n)|$. Then

$$\rho^{-n} u(n) \leq K |u(0)| + KL \sum_{k=0}^{n-1} \omega(k)$$

Notice that $\omega(n) = \sum_{j=1}^{\ell} \rho^{-p_j} \rho^{(-n-p_j)} |u(n - p_j)| \leq \rho^{-p} \sum_{j=1}^{\ell} v(n - p_j)$. For $n \geq 0$,

$$v(n) \leq Kv(0) + KL \sum_{k=0}^{n-1} \omega(k).$$

Let $z_n = \max\{|v(m)| : m = -p, -p+1, \dots, n\}$. Then, $\omega(n) \leq \rho^{-p\ell} z_n$, for all $n \geq 0$. Hence,

$$v(n) \leq Kv(0) + KL\rho^{-p\ell} \sum_{k=0}^{n-1} z_k.$$

Let $m_n \in \{-p, n-p+1, \dots, n\}$ such that $z_n = v(m_n)$.

If $m_n \geq 0$, then $z_n \leq Kv(0) + KL\rho^{-p} \sum_{k=0}^{m_n-1} z_k$. Hence,

$$z_n \leq Kv(0) + KL\ell\rho^{-p} \sum_{k=0}^{n-1} z_k.$$

If $m_n < 0$, then there is $j_0 \in \{1, \dots, p\}$ such that $m_n = n - j_0$. Since $K \geq 1$, $z_n \leq Kz_0$. So,

$$z_n \leq Kz_0 + KL\ell\rho^{-p} \sum_{k=0}^{n-1} z_k,$$

for all $n \geq 0$. By Gronwall's inequality,

$$z_n \leq (1 + K L \ell \rho^{-p})^n z_0.$$

So, for all $n \geq 0$,

$$|c(n) - \tilde{c}(n)| \leq K \rho^n (1 + K L \ell \rho^{-p})^n \max_{j=0,1,\dots,p} |c(-j) - \tilde{c}(-j)|. \quad (32)$$

By Lemma 5, there is a positive constant K_0 such that $|X(t, u)| \leq \sqrt{q} K_0$, for all $u \in J_{k(t)}$ and $|Z(t, \gamma^0(t))| \leq \sqrt{q} K_0 (1 + \sqrt{q} K_0 |B|_\infty \theta)$ for all $t \geq t_0$. By relation (29), for $t \geq t_0$, there is a positive constant K' such that

$$|y(t) - \tilde{y}(t)| \leq K' |c(n) - \tilde{c}(n)|. \quad (33)$$

The recent inequality show a Lipschitz continuous relation $\tilde{c} \mapsto \tilde{y}$.

By combining (32) and (33), this result is proved with $\tilde{K} = K' K$.

Notice that (31) implies (28). Then Theorem 3 insures the existence of the unique almost periodic solution of (2) which is exponentially stable. In fact, let $\alpha = \rho(1 + K L \ell \rho^{-p})$. By (31), $\alpha < 1$. For $\varepsilon > 0$ consider $\delta = \frac{\varepsilon}{K}$. By (30), (26) is satisfied and y is exponentially stable. \square

In the last theorem, the condition (31) is simple and slightly stronger than the condition of existence (28).

5. EXAMPLES

5.1. Exponential Dichotomy. It is not obvious to extend the exponential dichotomy for the difference equation (10) for the DEPCAG (4). We could consider an intuitively direct definition given by the existence of a projection Π_* and positive constants M and α such that

$$\begin{aligned} |Z(t, t_0) \Pi_* Z(s, t_0)^{-1}| &\leq M e^{-\alpha(t-s)}, \quad \text{if } t \geq s \\ |Z(t, t_0) (I - \Pi_*) Z(s, t_0)^{-1}| &\leq M e^{\alpha(t-s)}, \quad \text{if } t \leq s. \end{aligned} \quad (34)$$

However, if we take $A(t) = 0$, $B(t) = \text{diag}(\lambda_0(t), \lambda_1(t))$, $\lambda_0(t) = -\frac{2}{\pi} + \sin(2\pi t)$, $\lambda_1(t) = -\lambda_0(t)$, $t_n = n$ for all $n \in \mathbb{Z}$, $\int_n^{n+\delta} \lambda_0(\xi) d\xi = -\frac{1}{2\pi} (4\delta - 1 + \cos(2\pi\delta))$ and $\int_n^{n+\delta} \lambda_1(\xi) d\xi = \frac{1}{2\pi} (4\delta - 1 + \cos(2\pi\delta))$ for all $\delta \in [0, 1]$ then the exponential dichotomy on the difference equation (10) which can be written as (13) is satisfied for $\Pi = \text{diag}(1, 0)$ but there is no Π_* such that condition (34) is satisfied.

Notice that a dichotomy condition on the ordinary differential equation (6) implies an exponential dichotomy on the difference equation (10) [20, Proposition 2] when $|B(t)|$ is small enough and $y_\gamma(t) = y([t])$. However, an exponential dichotomy for the difference equation on (10) is not a necessary condition for an exponential dichotomy for the ordinary differential system (6). In fact, let's consider $t_n = n$, $A(t) = 0$ and $B(t) = \text{diag}(-\frac{3}{2}, \frac{1}{2})$. Then the exponential dichotomy for difference system (10) is satisfied, with no exponential dichotomy for the ordinary differential system (6).

5.2. Constant Coefficients. Assume that in DEPCAG (2), $A(t) = A_0$ and $B(t) = B_0$ are constants matrices and $F(t, \cdot)$ is almost periodic. Then DEPCAG (2) becomes

$$y'(t) = A_0 y(t) + B_0 y(\gamma^0(t)) + F(t, y_\gamma(t)), \quad t \in \mathbb{R}. \quad (35)$$

Assume that $t_{n+1} - t_n = \nu$ constant, A_0 and

$$H(n) = H_0 = e^{\nu A_0} [I + A_0^{-1}(I - e^{-\nu A_0})B_0]$$

are invertible.

By using $\sigma(H_0)$ as the usual notation for the spectrum of the matrix H_0 , it can be said that:

If $\sigma(H_0) \cap \{z \in \mathbb{C} : |z| = 1\}$ is the empty set and L in (5) satisfies (28), then DEPCAG (35) has an almost periodic solution. In particular, it is obtained when the elements of $\sigma(A_0)$ have non zero real part and $|B_0|$ is small enough.

If $\sigma(H_0) \subseteq \{z \in \mathbb{C} : |z| < 1\}$ and L in (5) satisfies (31), then DEPCAG (35) has an almost periodic solution which is exponentially stable. In particular, it is obtained when the elements of $\sigma(A_0)$ have negative real part and $|B_0|$ is small enough.

If $A_0 = 0$, $H(n) = I + \nu B_0$ invertible. If $\sigma(B_0) \subseteq \{z \in \mathbb{C} : |z| < 1/\nu\}$ is the empty set and L in (5) satisfies (28), then DEPCAG (35) has an almost periodic solution. If $\sigma(B_0) \subseteq \{z \in \mathbb{C} : |z| < 1/\nu\}$ and K in (5) satisfies (31), then DEPCAG (35) has an almost periodic solution which is exponentially stable. We can notice that it behaves as a difference equation.

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SAMUEL CASTILLO. DEPARTAMENTO DE MATEMÁTICA. FACULTAD DE CIENCIAS. UNIVERSIDAD DEL BÍO-BÍO. CASILLA 5-C. CONCEPCIÓN. CHILE.
E-mail address: `scastill@ubiobio.cl`

MANUEL PINTO. DEPARTAMENTO DE MATEMÁTICA. FACULTAD DE CIENCIAS. UNIVERSIDAD DE CHILE. CASILLA 653. SANTIAGO. CHILE.
E-mail address: `pintoj@uchile.cl`